1 Deriving (14.5) from the equations for the resonance curve of a harmonic oscillator

Equation (14.5) was derived without the assumption of a harmonic oscillator. However, it should be valid also for the harmonic oscillator, which is shown in the following.

In the footnote on page 235 it is mentioned that the equations for the resonance curve of a harmonic oscillator:

... (2.32) and (2.35) can be rearranged analytically leading to (14.5) in a very good approximation.

In order to show this, we start from (2.32) and (2.34):

$$\left(\frac{A}{A_{\text{drive}}}\right)^2 = \frac{1}{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2} \tag{1}$$

and

$$\tan\phi = \frac{-\omega\omega_0}{Q\left(\omega_0^2 - \omega^2\right)}. \tag{2}$$

Replacing

$$\left(\frac{\omega}{\omega_0}\right)^2 = x \quad \text{and} \quad \frac{1}{Q^2} = y \tag{3}$$

results in

$$\left(\frac{A}{A_{\text{drive}}}\right)^2 = \frac{1}{(1-x)^2 + xy} \tag{4}$$

and

$$\tan^2\phi = \frac{xy}{(1-x)^2}. \tag{5}$$

As

$$\frac{1}{\sin^2\phi} = \frac{1}{\tan^2\phi} + 1 = \frac{(1-x)^2}{xy} + 1 = \frac{(1-x)^2 + xy}{xy} = \frac{A}{\left(\frac{1}{A_{\text{drive}}}\right)^2 xy}, \tag{6}$$

the following equation holds

$$\sin^2\phi = \left(\frac{A}{A_{\mathrm{drive}}}\right)^2 xy. \tag{7}$$

Replacing x and y according to Eq. 3 results in

$$\sin(-\phi) = \frac{A}{A_{\mathrm{drive}}}\frac{1}{Q}\frac{\omega}{\omega_0}. \tag{8}$$

As $A_{\mathrm{free}} = QA_{\mathrm{drive}}$ finally the following equation arises

$$A = A_{\mathrm{free}}\sin(-\phi)\frac{\omega_0}{\omega}. \tag{9}$$

As $\omega \approx \omega_0$, close to the resonance, $A = A_{\mathrm{free}}\sin(-\phi)$, which corresponds to (14.5). Figure 13.11 shows for the case of a relatively small Q-factor ($Q = 100$) that $\omega \approx \omega_0$ close to the resonance. For larger Q-factors $\omega \approx \omega_0$ is even fulfilled better.

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